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Persistent Currents in Mesoscopic Rings: A Stochastic Model

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A stochastic model, proposed first by Landauer and Büttiker to explain the phenomenon of persistent currents in submicrometer normal metal rings, is developed quantitatively by determining the relevant relaxation time scales. The current excited by a periodically modulated magnetic field threading the ring is computed as the sum of two clear-cut components: a persistent current and a driven current. The latter component provides a notable example of a *stochastic resonance* mechanism in solid-state physics.

KEY WORDS: mesoscopics, persistent currents, stochastic resonance.

1. INTRODUCTION

In the presence of a static magnetic field, an isolated normal metal ring with linear dimensions smaller than the electron-phase coherence length was predicted(1,2) to bear an equilibrium *persistent* current which is periodic in the magnetic flux threading the loop. At zero temperature the characteristic length scales are the ring circumference $L \equiv 2\pi R$ and the cross-sectional area $A \equiv \pi w^2$ (the average width w is assumed to be much smaller than the loop radius R, $w \ll R$), the electron mean free path l, the localization length L_ξ, and the electron phase-coherence length L_ϕ. A conducting ring is termed *mesoscopic* when $L \leqslant L_\phi$. The order of magnitude of the characteristic length scales may vary depending on the material and the temperature.(4,5) Typically, for a pure Au ring(5) $L \sim 2\ \mu$m, $w \sim l \sim 0.07$ Å, and both L_ξ and L_ϕ are in excess of 10 μm at 40 mK.

The phenomenon of persistent currents, though confirmed experimentally,(4,5) has been puzzling to theorists.(6–11) One reason for controversy

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involves the very meaning of *averaging over disorder* (i.e., the impurity configurations) $\langle \cdots \rangle_D$: in fact, such an average can be carried out either for a constant number of electrons N (canonical average) or at constant chemical potential μ (grand-canonical average). The experimental observations were taken under diverse conditions: first,[4] on an ensemble of 10^7 disconnected rings with the same geometry but different number of electrons and impurity configuration; later,[5] on single isolated rings. The magnitude and the flux periodicity of the observed persistent currents depend crucially on the kind of averaging involved (e.g., on both N and disorder in the experiment ref. 4, none in the experiment of ref. 5).

Although the quantitative agreement between experimental observations and theoretical predictions is still far from satisfactory, a few facts are now well established. The disorder-averaged persistent current $I_P(\Phi) \equiv \langle I(\Phi) \rangle_D$ is a periodic function of the *constant* magnetic flux Φ threading the ring,

$$I_P(\Phi) = \sum_m I_m \sin\left(2\pi m \frac{\Phi}{\Phi_0}\right) \tag{1}$$

where $\Phi_0 = h/e$ denotes the magnetic-flux quantum in units with $c = 1$.

(i) Calculations for the one-channel loop and numerical simulations for the multichannel rings[6] indicated that averaging over disorder at constant N is different from averaging at constant μ, i.e., $\langle I(\Phi, N) \rangle_D \neq \langle I(\Phi, \mu) \rangle_D$. Moreover, averaging over N makes further averaging over disorder ineffective, $\langle I \rangle_N = \langle I \rangle_{N,D}$, and accounts for the period halving[4] of the function $I_P(\Phi)$ in Eq. (1), i.e., $\langle I_m \rangle_N = 0$ for m odd and $\langle I_m \rangle_N > 0$ for m even. In view of these results, $\langle I_P(\Phi) \rangle_N$ is a *paramagnetic* current, whereas the sign of I prior to disorder averaging is not defined. The difference between canonical and grand-canonical averages in mesoscopic ensembles recently has been given a solid theoretical ground.[9]

(ii) In the *ballistic* regime, $L \ll l$, $\langle I_{2m} \rangle_N = 2I_0/\pi m$ is independent of the number of transverse channels $M = Ak_F^2/4\pi$. Here I_0 is the natural unit for the current in a single-channel loop, $I_0 = ev_F/L$, where v_F is the electron velocity at the Fermi level, $\varepsilon_F = \hbar^2 k_F^2/2m$. In the *diffusive* regime, $l \ll L$, instead, $\langle I_{2m} \rangle_N$ scales with $1/M$.[7-9]

(iii) The single-electron description of the phenomenon underestimates the magnitude of the persistent currents in both experiments[4,5] by one order of magnitude or more.[6,9] This fact leads to the conclusion that the electron–electron interactions would account for the dominant contribution to the measured currents.[10]

In this paper we discuss *quantitatively* most of the results outlined above by means of a simple stochastic model originally proposed by

Landauer and Büttiker[2] to justify the persistency of nonvanishing equilibrium currents in isolated normal metal rings threaded by a constant magnetic field. Such a model proves, indeed, a useful tool to investigate the phenomenon of persistent currents in the *single-electron approximation.*

2. THE STOCHASTIC MODEL

The single-electron states of an isolated mesoscopic ring are described by the level structure of the periodic potential $U_D(x)$. For a single-channel loop, $M=1$, $U_D(x)=U_D(x+L)$ denotes the effective potential experienced by a single electron revolving around the loop circumference with a *static* configuration of *elastic* impurities with $l \ll L \leqslant L_\phi$. The vector potential associated with the magnetic field can be eliminated from the relevant Schrödinger–Bloch equation by modifying the natural periodicity of the nth eigenfunction $\psi_n(x)$,

$$\psi_n(x+L)=\exp\left(2\pi i \frac{\Phi}{\Phi_0}\right)\psi_n(x) \tag{2}$$

The corresponding energy level $\varepsilon_n(\Phi)$ is a periodic band in the flux Φ with period Φ_0. For a time-independent flux we obtain the zero-temperature current

$$I=\sum_n i_n=-\sum_n \frac{\partial \varepsilon_n}{\partial \Phi} \tag{3}$$

The summation in Eq. (3) is meant to include all the occupied states of $U_D(x)$ up to the Fermi energy (for the time being we ignore the spin degeneracy $s=2$). Although the actual band structure $\varepsilon_n(\Phi)$ is unknown, it can be proved[12] that the M transverse channels are degenerate in the ballistic regime, while a band splitting mechanism becomes effective in the diffusive regime. However, for weak disorder $l \ll L$, the resulting non-degenerate bands retain memory of their geometric degeneracy in the form of a strong statistical Φ correlation among a fraction M_{eff} of them. Successive higher-lying bands contribute with alternating signs to the current (3). Moreover, the current i_n increases with the energy, i.e., with n, so that at zero temperature and for $M=1$ the overall sign of I shall be determined by the highest-lying occupied band.[1,2] Its average with respect to disorder yields immediately $I_P(\Phi)$. In the multichannel case, instead, a summation over the M_{eff} topmost bands is required.[12]

As a further ingredient, we assume that *inelastic* scattering, no matter what the source, is weak enough to be treated as a *random* perturbation. If kT is smaller than the spacing of two bands adjacent to the Fermi level, electronic transitions between this pair of bands (i.e., between an occupied and an empty band) may occur as an effect of the inelastic (or electron-phase-breaking) processes.

Let us specialize the approach outlined above to the *diffusive* regime. To be more quantitative, we introduce a stochastic variable $E(t)$ for each of the $M_{\rm eff}$ contributing channels. The energy $E(t)$ is free to fluctuate between two metastable states $\varepsilon_{\pm}$ symmetrically located with respect to the Fermi level, $\varepsilon_{\pm} = \varepsilon_{\rm F} \pm \Delta(\Phi)/2$ (after disorder averaging). The two bands are separated by a *large* activation barrier, which can be overcome only in the presence of inelastic scattering. For simplicity, we assume that the energy fluctuations representing the inelastic effects are well described by a delta-correlated Gaussian noise, $\xi_E(t)$. Such an assumption follows the picture of electron phase breaking as the result of very many independent inelastic collisions. The *bistable* process $E(t)$ is now statistically determined if we can evaluate the intensity of $\xi_E(t)$, D_E, and the average switch time τ_R between $\varepsilon_{\pm}$.

The parameters D_E and τ_R are a function of the energy (length) scales of the problem. Let us consider an isolated gas of electrons in a impurity potential $U_D(x)$ and neglect the electron–electron interactions. The statistics of the energy spectrum $\varepsilon_n(\Phi)$ is characterized by[12]:

(i) The *mean spacing* of the electron energy bands adjacent to the Fermi level with respect to disorder and magnetic flux, $\Delta_0 \equiv \langle \Delta(\Phi) \rangle_{\Phi}$. The spacing Δ_0 is related to the electron density of states ν by the zeroth-order equality[3]

$$\nu = \frac{1}{L^d \Delta_0} \tag{4}$$

where $d=1$ is the effective dimensionality of the ring. The spacing Δ_0 is insensitive by definition to the actual disorder configuration and therefore cannot fully reproduce the transport properties of the conducting sample.

(ii) *The Thouless energy*[3,12] E_c, which quantifies the sensitivity of $\varepsilon_n(\Phi)$ to a change in the boundary conditions (2), i.e., $E_c \sim \partial^2 \varepsilon_n / \partial \Phi^2$. In the diffusive regime, E_c is the reciprocal of the time needed by a single electron to diffuse through the ring circumference L, i.e.,

$$E_c = \frac{\hbar D}{L^2} \tag{5}$$

where $D = l v_{\rm F}$ is the spatial diffusion coefficient for a one-dimensional electron system. It is well known[3] that D is sensitive to quantum interference, while Δ_0 is not. As a consequence, in the limit of interest, $l \ll L$, *weak localization* corrections affect the conductivity of the sample

$$\sigma = e^2 \nu D \tag{6}$$

but not the density of states ν.

A useful relationship between E_c and Δ_0 can be established for the electron energy spectrum in a normal metal ring. For a one-dimensional loop the mean spacing of the electronic bands at the Fermi level is hv_F/L. In the presence of M transverse channels, instead, $\Delta_0 = hv_F/LM$, whence, by substituting D with lv_F in Eq. (5),

$$E_c = \Delta_0 \frac{Ml}{L} \equiv \Delta_0 M_{\text{eff}} \tag{7}$$

As anticipated, disorder removes the geometric degeneracy due to the M transverse channels, a fraction $M_{\text{eff}} = Ml/L$ of which retains a strong statistical correlation. In the presence of strong disorder, we have Anderson localization and $M_{\text{eff}} \to 1$, i.e.,

$$L_\xi = Ml \tag{8}$$

We recall that in the case under study, $l \ll L \ll L_\xi$ and M is typically of the order of 10^2 or larger.[4,5]

We are ready now to calculate D_E and τ_R. In the presence of inelastic processes the periodicity of the system, Eq. (2), is broken. We can imagine the motion of a single electron as if it were free to scatter elastically over a distance of the order of L_ξ, which it diffuses in a time interval τ_ξ given by $L_\xi^2 = (Ml)^2 = D\tau_\xi$, that is,

$$\tau_\xi = \frac{\hbar}{\Delta_0} M_{\text{eff}} \tag{9}$$

Such a mechanism allows M_{eff} electrons at the Fermi level to jump independently between adjacent energy bands within an energy range E_c, which, in fact, represents the maximum sensitivity of $\varepsilon_n(\Phi)$ to the breach of the periodic boundary conditions caused by inelastic scattering. This implies that each level $\varepsilon_n(\Phi)$ within the sensitivity energy range fluctuates with a diffusion coefficient D_E given by $E_c^2 = D_E \tau_\xi$, whence

$$D_E = \frac{\Delta_0^2 E_c}{\hbar} \tag{10}$$

D_E is by definition the intensity of the random noise $\xi_E(t)$ driving the stochastic observable $E(t)$.

In order to estimate τ_R, we remark that the dephasing $\Delta\phi$ in the electron wave function caused by a jump between two adjacent bands increases with time according to the law $\Delta\phi(t) = \Delta_0 t/\hbar$. On the other hand, we know that the dephasing due to inelastic scattering is characterized by the length scale $L_\phi \gg L$. Therefore, the electron phase breaking through the loop circumference in the presence of the inelastic effect is of the order of L/L_ϕ. The average switch time τ_R is determined by the identity $\Delta\phi(\tau_R) \equiv L/L_\phi$, i.e.,

$$\tau_R = \frac{\hbar}{\Delta_0}\left(\frac{E_c}{E_\phi}\right)^{1/2} = \frac{\hbar}{\Delta_0}\frac{L_\phi}{L} \tag{11}$$

where we made use of the definition $E_\phi \equiv \hbar D/L_\phi^2$.

Let us summarize our stochastic model for the persistent currents in disordered normal metal rings. The disorder average of the current (3) is determined by the additive contributions of the M_{eff} topmost occupied bands. The conducting electrons are characterized each by a fluctuating energy $E(t)$ which switches with rate $1/\tau_R$ between the energy values of two bands, one occupied and one empty, adjacent to the Fermi level. Energy fluctuations are the effect of independent inelastic collisions, represented here by a white Gaussian noise acting upon $E(t)$, with intensity D_E.

3. CURRENTS INDUCED IN A MESOSCOPIC RING BY A THREADING MAGNETIC FLUX

We apply the model outlined in the previous section to the problem of the stationary currents induced in a weakly disordered normal metal ring by a periodically modulated magnetic flux threading the loop, e.g.,

$$\Phi(t) = \Phi_{\text{DC}} + \Phi_{\text{AC}} \cos \Omega t \tag{12}$$

The static case, $\Phi_{\text{AC}} = 0$, has received a lot more attention by theorists[7-9] puzzled by the very nature of the phenomenon under investigation. The oscillating case[11] is of practical interest, too, since the experimental observations make use of a sinusoidal flux (12) with $\Phi_{\text{AC}} \ll \Phi_{\text{DC}}$ in order to single out the Φ dependence of the monitored currents relative to strong noise backgrounds,[4,5] according to the expansion

$$I_P(\Phi) = I_P(\Phi_{\text{DC}}) + \left.\frac{\partial I_P}{\partial \Phi}\right|_{\phi_{\text{DC}}} \Phi_{\text{AC}} \cos \Omega t + \cdots \tag{13}$$

Of course, a periodic magnetic field has the further effect of stimulating a periodic current consistent with the requirements of the linear response theory,

$$I_\sigma = \sigma(\Phi_{\mathrm{DC}})\, \aleph(t) \tag{14}$$

where $\aleph$ denotes the induced electromotive force $(1/L)\, d\Phi/dt$.

a. *Persistent currents in the static case.* As anticipated in Section 2, the electron energy bands are strongly correlated in pairs. To make that argument more quantitative, let us consider a pair $(n, n+1)$ of bands adjacent to the Fermi level for a given impurity configuration. On applying the Brillouin–Wigner expansion of ref. 6, one finds

$$\varepsilon_n(\Phi_{\mathrm{DC}}) = \sum_{p=0}^{\infty} \lambda_n^{(p)} \cos\left(2\pi p \frac{\Phi_{\mathrm{DC}}}{\Phi_0}\right) \tag{15}$$

where the coefficients $\lambda_n^{(p)}$ obey the following approximate equalities:

$$\lambda_{n+1}^{(1)} = -\lambda_n^{(1)}$$
$$\lambda_{n+1}^{(2)} = -\lambda_n^{(2)} = -(\lambda_n^{(1)})^2/[\varepsilon_{n+1}^{(0)} - \varepsilon_n^{(0)}]$$

and $\varepsilon_n^{(0)}$ are the unperturbed electron energy levels. When averaging over the number of electrons N, the current contributions $i_n(\Phi)$ tend to cancel out pairwise. The sign of the first harmonic in Eq. (15) depends on n, while the second harmonic is always nonnegative for the lower energy level. Correspondingly, the first harmonic of the averaged current $\langle I_P(\Phi)\rangle_N$ vanishes, while the second harmonic is positive definite. This argument can be extended to affirm that all the odd harmonics of $\langle I_P(\Phi)\rangle_N$ are vanishingly small, whence the *period halving* observed in the experiment of ref. 4.

Most notably, in the diffusive regime, $\lambda_n^{(p)}$ is proportional to $W^p M_{\mathrm{eff}}$, where W is the probability for the electron to diffuse in a one-dimensional sample through a distance equal to the loop circumference L, the number of transverse channels M_{eff} plays the role of path degeneracy, and p is the winding number, telling us how many times the magnetic flux Φ has been encircled by the diffusing electron. As a further complication, we have introduced explicitly the inelastic effects, which determine a finite lifetime τ_R of the electronic states with periodic boundary conditions (2). This implies that the probability W decays exponentially over one revolution time L^2/lv_{F}, i.e., $W \to W \exp(-L^2/lv_{\mathrm{F}}\tau_R)$. In conclusion, we obtain

$$\langle I_P(\Phi_{\mathrm{DC}})\rangle_N = \sum_m c_m \exp\left[-2m\left(\frac{E_\phi}{E_c}\right)^{1/2}\right] \sin\left(4\pi m \frac{\Phi_{\mathrm{DC}}}{\Phi_0}\right) \tag{16}$$

with $c_m = (4\pi m/\Phi_0)\langle\lambda_n^{(2m)}\rangle_N$. The explicit calculation of $\langle\lambda_n^{(2m)}\rangle_N$ lies beyond the reach of our simple model. However, the coefficients c_m can be determined by making contact with Eq. (33) of ref. 12: on posing $\langle\varepsilon_{n+1}(\Phi) - \varepsilon_n(\Phi)\rangle_N = \Delta(\Phi)$ and inserting Eq. (6), one concludes that

$$c_m = \frac{2\Delta_0}{\pi\Phi_0} \tag{17}$$

The summation (16) can be performed explicitly to yield

$$\langle I_P(\Phi_{\mathrm{DC}})\rangle_N = \frac{2\Delta_0}{\pi\Phi_0}\frac{p\sin q}{1-2p\cos q+p^2} \tag{18}$$

with $p = \exp[-2(E_\phi/E_c)^{1/2}]$ and $q = 4\pi\Phi_{\mathrm{DC}}/\Phi_0$. Including the spin degeneracy $s=2$ amounts to multiplying the rhs of Eqs. (16) and (17) by a factor s^2. Alternatively, for a generic static flux Φ, $\langle I_P(\Phi)\rangle_N$ can be written as

$$\langle I_P(\Phi)\rangle_N = -\frac{\partial\Delta(\Phi)}{\partial\Phi} \tag{19}$$

$$\text{with } \Delta(\Phi) = -\frac{\Delta_0}{4\pi^2}\ln(1-2p\cos q+p^2) \tag{20}$$

The results (16) and (17) were obtained first in ref. 9.

b. *Currents driven by an oscillating magnetic flux.* The current $I_\sigma(t)$ driven by the oscillating component of $\Phi(t)$, Eq. (12), can be easily computed within the framework of the stochastic model of Section 2. The time-dependent magnetic flux induces an electromotive force $\aleph = -(\Omega/L)\,\Phi_{\mathrm{AC}}\sin\Omega t$. The fluctuating energy $E(t)$ is then driven by an effective forcing term of the order of the dissipated power

$$-\langle e\aleph\cdot\mathbf{v}_{\mathrm{F}}\rangle_D = \frac{eD}{L}\,\Omega\Phi_{\mathrm{AC}}\sin\Omega t \tag{21}$$

In Eq. (21) the angular average brings in a factor l/L due to the assumed dissipative regime for the electronic motion. The variable $E(t)$ denotes, then, a bistable stochastic process driven by a periodic force (21). Such a process will exhibit *stochastic resonance.*[13–15] In particular, the stochastic average $\langle E(t)\rangle$ for a *single*-channel loop can be obtained from Eqs. (7) of ref. 15 with the following substitutions: $x \to E$, $x_m \to \Delta(\Phi_{\mathrm{DC}})/2$, $D \to D_E$, $\lambda \to 1/\tau_R$, and $a \to eD\Omega\Phi_{\mathrm{DC}}/L^2$. If, for simplicity, we agree to take the further average of $\langle E(t)\rangle$ over Φ_{DC}, we obtain

$$\langle E(t)\rangle_\Phi = E_0\sin(\Omega t + \alpha) \tag{22}$$

with

$$\operatorname{tg} \alpha = -\Omega\tau_R \tag{23}$$

and

$$E_0 = \frac{eD}{L^2}\frac{\hbar\Omega}{E_c}\frac{\Phi_{AC}}{[1+(\Omega\tau_R)^2]^{1/2}} \tag{24}$$

In the presence of M transverse channels, E_0 has to be multiplied by M_{eff}, the number of statistically correlated channels. The driven current associated with $\langle E(t)\rangle_\phi$ follows immediately,

$$\langle I_\sigma(t)\rangle_\Phi = \frac{eD}{L^2}\frac{\hbar\Omega}{E_c}\frac{\Phi_{AC}}{\Phi_0}\frac{1}{[1+(\Omega\tau_R)^2]^{1/2}}\sin(\Omega t+\alpha) \tag{25}$$

Note that in view of Eqs. (6) and (21) with $\langle \Delta(\Phi_{DC})\rangle_\Phi = \Delta_0$, the amplitude of $I_\sigma(t)$ reads

$$\frac{\sigma}{[1+(\Omega\tau_R)^2]^{1/2}}\,|\aleph| \tag{26}$$

as expected from the linear response theory.

The amplitude of $\langle E(t)\rangle_\Phi$ might exhibit a typical stochastic resonance behavior due to the fact that $1/\tau_R$ tends to vanish at zero temperature, i.e., in the absence of inelastic processes. However, in the experimental conditions of refs. 4 and 5, $\Omega \simeq 10^2$ Hz and $\Omega\tau_R \sim 1$ for $\Omega \sim 10^8$ Hz. This means that in the analysis of the relevant measurements, the driven current $I_\sigma(t)$ may be safely neglected compared to $I_P(\Phi_{DC})$, contrary to the conclusions of ref. 11.

4. CONCLUSIONS

The stochastic model we developed for investigating the phenomenon of persistent currents in mesoscopic rings confirms the predictions of both the Green-function diagrammatic technique and the linear response theory. The advantage offered by such a model is the simple intuitive picture of the phenomenon presented in Section 3.

Any sensible approach to this puzzling problem should be improved to go beyond the single-electron approximation. At the present time, it sounds reasonable to conclude that in a *quasi-one-dimensional* ring threaded by an external magnetic flux the measured persistent currents are mostly due to the electron–electron interactions.

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